

fall ; for which he had little or no care taken of him. He has ever since this accident had some complaints in his side at times, but not constantly ; nor have they ever been so bad, as to prevent his acting in his business as a sailor, till within a few weeks before he applied to me.

London, Hatton Garden,
June 28, 1753.

*XL. Extract of a Letter from Mr. James
Dodson to Mr. William Mountaine, F.R.S.*

May 26, 1753.

Read July 5, 1753. **T**HE world has, without dispute, been obliged to the invention of fluxions, for many concise methods of calculating the peripheries, areas, and solidities, of curvilinear figures ; but it must be confessed, at the same time, that the most useful, even of those, had been computed before, tho' by methods more laborious ; and, consequently, since the truth of the principles of fluxions was long disputed, that art seems rather to have received, than to have afforded, any advantage, in those cases.

Neper and Briggs calculated their several tables of logarithms, with almost insuperable labour ; and Van Ceulen was rendered famous for his approximation to the quadrature of the circle, on account of the acknowledged tediousness of its computation. The methods of computing logarithms were indeed improved, by the assistance of the properties of the Hyperbola, and

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the use of infinite series, before the great inventor of fluxions was known in the world : but then, the business of logarithms being purely arithmetical, the Hyperbola was foreign to the subject ; and the nature of infinite series, tho' well adapted to the purpose, was at that time but little understood.

I am sometimes induced to believe, that, if the latter (*i. e.* the nature and use of infinite series) had arrived at any degree of perfection, before the invention of fluxions, most of the series, which are given for the above-named kind of calculations, and have been deduced from fluxional processes, would have been discovered without the assistance of them : and I am of this opinion, because I am certain, that many of them might have so been : to instance in both the cases above quoted ; *viz.* the series, for making logarithms, and for rectifying the circle.

And first, the terms of one of the most simple series, for expressing the logarithm of a given number, is composed of the powers of the excess of that number, above unity, divided by their respective indices ; of which the first, third, fifth, &c. terms are affirmative, and the second, fourth, sixth, &c. terms are negative ; and the difference between the sums of the affirmative and the negative terms, is the Neperian, hyperbolic, or (as some call it) the natural logarithm of the given number.

Now a mathematician, who understands the nature and management of series (altho' wholly ignorant of fluxions, or what the justly celebrated Dr. Halley, in his admired investigation of this very series, published in N^o 216 of the *Philosophical Transactions*, calls *ratiunculae*, &c.) might arrive at the same conclusion, in the following manner :

Since

Since the logarithm of unity is universally determined to be nothing; that of 2, 3, 4, 10, or any other number, consider'd as a root, is one; that of 4, 9, 16, 100, &c. consider'd as the square of that root, is two; and so on; it follows, that (in all cases) the logarithm of a greater number will exceed that of its lesser; and each logarithm will have some relation to the excess of its number, above unity, the number, whose logarithm is nothing: the terms of the series, therefore, which will represent the logarithm of any number, will consist of the powers of the excess of that number, above unity, with some, yet unknown, but constant coefficients.

That the logarithm of the square of any number is twice the logarithm of its root, is a well-known property of those artificial numbers; and therefore, the doubles of the particular terms of the assumed series will constitute a series, expressing the logarithm of the square of the given number.

But, by the fourth proposition of the second book of Euclid, the square of any quantity is equal the sum of the squares of its two parts, more a double rectangle of those parts; which, in this case (where the given number has been assumed, to consist of unity, and an excess) will be unity, more twice that excess, more the square thereof.

If, therefore, the several powers of the compound quantity (twice the excess of the given number above unity, more the square thereof) be multiplied by the above assumed coefficients, and afterwards ranged under each other, according to the powers of the said excess; their sums will again express the logarithm of the square of the given number.

Now, since the logarithm of the square of the given number may be thus expressed by two infinite series, each constituted of the excess thereof, above unity, and its powers; it follows, that the coefficients of the like powers of that excess, in each series, will be equal between themselves; and, consequently, the values of the unknown coefficients may be obtained, by simple equations; and these coefficients will, by the process annexed, appear to be, the reciprocals of the several indexes of the powers of that excess, affected alternately with the signs $+$ and $-$, as was before found, by the quadrature of the Hyperbola, by Dr. Halley in the above-cited *Philosophical Transaction*, and by many who have used a fluxional process.

But, there is another logarithmic series equally simple with the former, consisting of the same terms, but all affirmative. This has been demonstrated to be the logarithm of that fraction, whose numerator is unity, and denominator a number, as much less than unity, as the former number exceeded it.

Now, if an infinite series be formed from that fraction, by actual division, it will consist of unity, and all the powers of that defect; and if the several powers of the excess of this infinite series above unity, be multiplied by the coefficients above-found, and ranged according to the powers of that defect, their sums will exhibit the above-described series for the logarithm of that fraction, as appears by the operation subjoin'd.

As to the application of these two series, and their sum, to the finding the logarithms of numbers; the same, being copiously treated of by Dr. Halley (in the *Philosophical Transaction* before quoted) there is

no occasion for the repetition thereof here. *Note,* The above tract of Dr. Halley's is printed with the explication of Sherwin's tables of logarithms (fol. 10) with many examples of the use thereof annexed.

Secondly, The terms of one of the best series, for the rectification of the circle, are composed of the odd powers of the tangent of any arc, not exceeding 45 degrees, severally divided by their respective indexes ; of which the first, third, fifth, &c. terms are affirmative ; and the second, fourth, sixth, &c. terms are negative ; and the difference, between the sums of the affirmative and negative terms, is the length of that arc, of which the tangent, and its powers, constitute the series.

Now a mathematician, who understands the nature and management of series, altho' wholly ignorant of fluxions, might investigate this series in the following manner :

It has been geometrically demonstrated, that, the radius of a circle being unity, if the double of the tangent of any arc, be divided by the difference between unity, and the square of that tangent, the quotient will be the tangent of twice the arc.

Now if an infinite series be formed by actual division, its terms will consist of the doubles of the odd powers of the tangent, and will be all affirmative ; which series will express the length of the tangent of the double of that arc, whose tangent and its powers constitute the same.

If a series, consisting of the tangent and its powers, with unknown coefficients, be assumed (as in the former case) to express the length of the arc ; then the length of the double of that arc may be expressed two ways ; *viz.* either by multiplying each term of the

the series assumed by the number two ; or by finding the powers of the series above described (which exhibits the length of the tangent of the double arc) multiplying each power by its proper coefficient, ranging the products under each other (according to the powers of the tangent of the single arc) and finding their sum.

Now, since the length of the double arc may be thus expressed, by two infinite series, each constituted of the tangent of the single arc, and its powers ; therefore the coefficients of the like powers of that tangent, in each series, will be equal between themselves ; and consequently the values of the unknown coefficients may be obtained by simple equations.

Lastly, since the series, which gives the length of the tangent of the double arc consists only of the odd powers of the tangent of the single arc, therefore none of the even powers thereof can range therewith : now these will not occur in the odd powers of that series ; and therefore the series, assumed to express the length of the single arc, whose double is to be compared with the sum of the former, must consist only of the odd powers of that tangent ; and then the series first mentioned results from the operation, as will appear by examining the same, as hereto annexed.

I am thoroughly sensible, that, to the learned, who are already masters of the investigation of these series by other methods, this will appear to be an essay of more curiosity than use ; but, with regard to students, I humbly conceive it will have its advantages ; because by those, who are acquainted with the notation of algebra, and the manner of solving
simple

simple equations, the nature of the series, and the operations above used, may be easily obtained; and they may be hereby enabled to make a scientific use of logarithms in arithmetical, and of the rectification of the circle, in geometrical calculations; which at present cannot be done, till a much greater progress is made in mathematical knowledge, without great labour; as well as to examine and correct the printed tables of both sorts, if dubious, or found to be erroneous.

The operation necessary to find the coefficients of a series, which will express the logarithm of a given number.

If the given number be represented by $1+n$, then the following series may be assumed to represent its logarithm:

$$n + xn^2 + yn^3 + zn^4 + un^5, \&c.$$

And $2n + 2xn^2 + 2yn^3 + 2zn^4 + 2un^5, \&c.$ will represent the logarithm of the square of that number; viz. of $1 + 2n + nn$.

But, because $2n+nn$ is the excess of $1 + 2n + nn$, above unity, therefore its logarithm will be also expressed by

$$2n+nn + x \times 2n+nn^2 + y \times 2n+nn^3 + z \times 2n+nn^4, \&c.$$

$$\begin{array}{lcl} \text{Now } \frac{2n+nn^2}{2n+nn^3} & = & 4nn + 4n^3 + n^4 \\ \frac{2n+nn^3}{2n+nn^4} & = & 8n^3 + 12n^4 + 6n^5, \&c. \\ \frac{2n+nn^4}{2n+nn^5} & = & 16n^4 + 32n^5, \&c. \\ & & 32n^5, \&c. \end{array}$$

Therefore,

$$\begin{aligned}
 \overline{2n+nn} &= 2n+nn; \\
 x \times \overline{2n+nn^2} &= 4xnn + 4xn^3 + xn^4 \\
 y \times \overline{2n+nn^3} &= 8yn^3 + 12yn^4 + 6yn^5 \&c. \\
 z \times \overline{2n+nn^4} &= 16zn^4 + 32zn^5 \&c. \\
 u \times \overline{2n+nn^5} &= 32un^5 \&c.
 \end{aligned}$$

And the sum of these is equal to the logarithm of the square of $1+n$.

If an equation be formed, of the coefficients of n^2 , in each of these expressions of the logarithm of that square,

Then $2x = 1 + 4x$; whence $-\frac{1}{2} = x$.

And, by proceeding in the same manner with the coefficients of n^3 , n^4 , n^5 , &c. and supplying the places of x , y , z , &c. as they arise, by the numbers (so found) we shall have

$$\begin{aligned}
 2y &= -\frac{4}{2} + 8y; & \text{whence } +\frac{1}{2} &= y; \\
 2z &= -\frac{1}{2} + \frac{1}{3} + 16z; & \text{whence } -\frac{1}{4} &= z; \\
 2u &= \frac{6}{3} - \frac{3}{4} + 32u; & \text{whence } +\frac{1}{7} &= u;
 \end{aligned}$$

Consequently, the logarithm of $1+n$ will be expressed by $n - \frac{1}{2}n^2 + \frac{1}{3}n^3 - \frac{1}{4}n^4 + \frac{1}{5}n^5$, &c. as above asserted.

Again, since $\frac{1}{1-n} = 1+n+n^2+n^3+n^4+n^5$, &c. as appears by the following division :

$$\begin{array}{r}
 1-n \) \ 1 \quad (1+n+n^2, \&c. \\
 \underline{1-n} \\
 + n \\
 \underline{+ n - nn} \\
 + nn
 \end{array}$$

And, since the excess of that series, above unity, is the series $n + n^2 + n^3 + n^4$, &c.

Therefore

Therefore the logarithm of $\frac{1}{1-n}$ will consist of the sums of the powers of that series, multiplied by the above-found coefficients $\frac{1}{1}, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4}, +\frac{1}{5}, \&c.$

$$\text{Now the } \left\{ \begin{matrix} 2 \\ 3 \\ 4 \\ 5 \end{matrix} \right\} \text{ Power of that Series will be } \left\{ \begin{matrix} n^2 + 2n^3 + 3n^4 + 4n^5, \&c. \\ n^3 + 3n^4 + 6n^5, \&c. \\ n^4 + 4n^5, \&c. \\ n^5, \&c. \end{matrix} \right.$$

And,

$$\left\{ \begin{matrix} 1 \\ -\frac{1}{2} \\ +\frac{1}{3} \\ -\frac{1}{4} \\ +\frac{1}{5} \end{matrix} \right\} \text{ of that } \left\{ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \right\} \text{ Power will be } \left\{ \begin{matrix} n + n^2 + n^3 + n^4 + n^5, \&c. \\ -\frac{1}{2}n^2 - \frac{1}{2}n^3 - \frac{1}{2}n^4 - \frac{1}{2}n^5, \&c. \\ +\frac{1}{3}n^3 + \frac{1}{3}n^4 + \frac{1}{3}n^5, \&c. \\ -\frac{1}{4}n^4 - \frac{1}{4}n^5, \&c. \\ +\frac{1}{5}n^5, \&c. \end{matrix} \right.$$

The sums of which, viz. $n + \frac{1}{2}n^2 + \frac{1}{3}n^3 + \frac{1}{4}n^4 + \frac{1}{5}n^5, \&c.$ will be the logarithm of $\frac{1}{1-n}$, as above affirmed.

The operation necessary, to find the coefficients of a series, which will express the length of the arc of a circle, by the tangent of that arc, and its powers,

Let a represent the length of the arc, and t the length of the tangent thereof; then the tangent of that arc whose length is $2a$, will be $\frac{2t}{1-tt}$; which fraction is equal to the infinite series, $2t + 2t^3 + 2t^5 + 2t^7 + 2t^9, \&c.$ See the division.

$$\begin{array}{r}
 [\quad 282 \quad] \\
 1-tt) 2t \quad (2t + 2t^3 + 2t^5, \&c. \\
 \underline{2t - 2t^3} \\
 \quad + 2t^3 \\
 \quad \underline{+ 2t^3 - 2t^5} \\
 \qquad \quad + 2t^5
 \end{array}$$

And by performing the necessary multiplications, or divisions, it will also appear, that

$$\left[\frac{2t}{1-tt} \right]^3 = 8t^3 + 24t^5 + 48t^7 + 80t^9, \&c.$$

$$\left[\frac{2t}{1-tt} \right]^5 = 32t^5 + 160t^7 + 480t^9, \&c.$$

$$\left[\frac{2t}{1-tt} \right]^7 = 128t^7 + 896t^9, \&c.$$

$$\left[\frac{2t}{1-tt} \right]^9 = 512t^9, \&c.$$

Now if we assume, for the value of a , the following series, $t + xt^3 + yt^5 + zt^7 + ut^9, \&c.$

Then $2t + 2xt^3 + 2yt^5 + 2zt^7 + 2ut^9, \&c. = 2a.$

And because $\frac{2t}{1-tt}$ is the tangent of the arc, whose length is $2a$, therefore

$$\frac{2t}{1-tt} + x \times \left[\frac{2t}{1-tt} \right]^3 + y \times \left[\frac{2t}{1-tt} \right]^5 + z \times \left[\frac{2t}{1-tt} \right]^7 \&c. = 2a.$$

Which expression is equivalent to the sum of the following series; for

$$\begin{aligned}\frac{2t}{1-tt} &= 2t + 2t^3 + 2t^5 + 2t^7 + 2t^9, \text{ \&c.} \\ x \times \frac{2t}{1-tt} &= 8xt^3 + 24xt^5 + 48xt^7 + 80xt^9, \text{ \&c.} \\ y \times \frac{2t}{1-tt} &= 32yt^5 + 160yt^7 + 480yt^9, \text{ \&c.} \\ z \times \frac{2t}{1-tt} &= 128zt^7 + 896zt^9, \text{ \&c.} \\ u \times \frac{2t}{1-tt} &= 512ut^9, \text{ \&c.}\end{aligned}$$

And (by making an equation, between $2x$, the coefficient of t^3 in the first found value of $2a$, and $2+8x$, the sum of the coefficients of t^3 in the latter) $2x = 2 + 8x$; whence $-\frac{1}{3} = x$.

And, by proceeding in the same manner, with the coefficients of t^5, t^7, t^9 , &c. and supplying the places of x, y, z , &c. (as they arise) by the numbers (so found) we shall have

$$\begin{aligned}2y &= 2 - \frac{24}{3} + 32y; & \text{whence } y &= +\frac{1}{4}. \\ 2z &= 2 - \frac{48}{3} + \frac{160}{7} + 128z; & \text{whence } z &= -\frac{1}{7}. \\ \text{And } 2u &= 2 - \frac{80}{3} + \frac{160}{7} - \frac{896}{7} + 512u; & \text{whence } u &= +\frac{1}{9}.\end{aligned}$$

Therefore we may conclude, that $t - \frac{1}{3}t^3 + \frac{1}{7}t^5 - \frac{1}{9}t^7 + \frac{1}{9}t^9$, &c. $= a$.

The application of this series to the rectification of the circle is extant in many authors; particularly in Sherwin's tables of logarithms, above-quoted, folio 55.

When the arc is just 45 degrees, then $t = 1$, and the series becomes $\frac{1}{4} - \frac{1}{3} + \frac{1}{7} - \frac{1}{9} + \frac{1}{9}$, &c. which converges exceedingly slow; but, by the assistance of a

method, given in the appendix to M. De Moivre's *Miscellanea Analytica*, it may be transformed to another, converging quicker; which method is applied to this very series, in folio 362 of the *Mathematical Repository*, Vol. I.

XLI. *A Letter from John Lining, M. D. of Charles-Town, South-Carolina, to the Rev. Thomas Birch, D. D. Secr. R. S. concerning the Quantity of Rain fallen there from January 1738, to December 1752.*

Rev. Sir,

South Carolina, Charles-Town, April
9, 1753.

Read July 8, 1753. **T**HE favourable reception, which my former papers met with from the Royal Society, encourages me to send you a table of the quantity of rain, which fell in Charles-Town for these 15 years last past; which, if continued for half a century, might be of use, in discovering to us the changes made in a climate, by clearing the land of its woods. Tho' I formerly sent a table of the rain from 1738 to 1745 inclusive, which is publish'd in N^o 487 of the *Philosophical Transactions*; yet, as I thought it would be more convenient to bring the whole into one view, I have not only added to this table the rain of those years, but have likewise distinguished the quantity which fell in the several seasons. In this table I continued the old stile to the first of last February, that the mean quantity, in each month, and in the different seasons, might be given exactly.

As